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Thermal Entanglement in Anisotropic Heisenberg XYZ Chain with External Magnetic Field at Any Finite *T*

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Abstract The thermal entanglement of a Two-qubit anisotropic Heisenberg XYZ chain in thermal equilibrium at temperature T in the presence of external magnetic field is investigated, the combined influence of anisotropic interaction and the inhomogeneous magnetic field on the thermal entanglement of the system is examined. Our results show that, the thermal entanglement can be produced at any finite T by adjusting the magnetic field strength, and the critical magnetic field ϵ for which the concurrence C vanishes is increased by introducing the interaction of the z-component of two neighboring spins J_z, and by the increasing of the anisotropic parameter γ.

Keywords Heisenberg model · Magnetic field · Anisotropic

1 Introduction

Entanglement is a nonlocal correlation between quantum systems that does not exist classically. Entangled pairs of quantum systems remain strongly correlated even if they are well separated spatially; Observing the state of one fixes with certainty the state of the other. In recent years, quantum entanglement has become recognized as crucial in various fields of quantum information, such as quantum cryptography [1], teleportation [2], and quantum computation [3]. Potential applications of entanglement in these fields have stimulated research on ways to quantify and control it. Considerable attention has been devoted to interacting Heisenberg spin systems, which may be used for gate operations in solid states quantum computation processors [4–6]. The thermal entanglement in an isotropic Heisenberg spin chain has been studied in the absence [7] and in the presence of an external magnetic field B [8–10]. They found that the entanglement of the two-qubit isotropic Heisenberg system decreases with increasing T and vanishes beyond a critical value T_c [8, 9], which is independent of B. The corresponding anisotropic case has been investigated in the case of

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$B=0$ [9] and $B\neq 0$ [11]. In Ref. [9], increasing the anisotropy is found to reduce monotonically the system's concurrence for any finite T and to decrease the critical temperature T_c beyond which the concurrence vanishes.

Recently the effect of inhomogeneous magnetic field on the thermal entanglement of an isotropic two-qubit XXX spin system has been studied in Ref. [12], including the interaction of z-component J_z of two neighboring spins. Reference [13] also studied the effect of inhomogeneous magnetic field on the thermal entanglement in a two-qubit Heisenberg XXZ spin chain. In this paper, we investigate the influence of anisotropy, interaction of z-component J_z and inhomogeneity of the external magnetic field the on the entanglement of a two-qubit Heisenberg XYZ chain at thermal equilibrium. We demonstrate that these parameters may together be used to control the entanglement at zero temperature and a finite temperature and, in particular, to produce entanglement at any finite T.

2 The Model and Theoretical Treatment

The Heisenberg Hamiltonian of a N-qubit anisotropic Heisenberg XYZ model with external magnetic fields is

$$H=\frac{1}{2}\sum_{i=1}^{N}[J_x\sigma_i^x\sigma_{i+1}^x+J_y\sigma_i^y\sigma_{i+1}^y+J_z\sigma_i^z\sigma_{i+1}^z+B_i\sigma_i^z], \tag{1}$$

where $(\sigma^x,\sigma^y,\sigma^z)$ are the vector of Pauli matrices and J_j $(j=x,y,z)$ is the anisotropic coupling coefficient between the nearest two spins. The parameter $J_j>0$ means that the chain is antiferromagnetic, and ferromagnetic for $J_j<0$. B_i is the local magnetic field at the ith qubit. The differential value of B_i-B_j controls the degree of inhomogeneity between qubit i and j. For a spin system in equilibrium at temperature T, the density matrix is $\rho=\frac{1}{Z}\exp(-H/k_BT)$, where H is the Hamiltonian of this system, Z is the partition function and k_B is the Boltzmann constant. Usually we write $k_B=1$. For a two-qubit system the thermal entanglement can be measured by the concurrence C which can be calculated with the help of Wootters' formula [14] $C=\max(0,2\max\lambda_i-\sum_{i=1}^{4}\lambda_i)$, where λ_i is the square roots of the eigenvalues of the matrix

$$R=\rho(\sigma_1^y\otimes\sigma_2^y)\rho^*(\sigma_1^y\otimes\sigma_2^y), \tag{2}$$

where the asterisk indicates complex conjugation, the concurrence C ranges from zero to one. Consider now the Hamiltonian H for the anisotropic two-qubit Heisenberg XYZ chain in an inhomogeneous magnetic field. The Hamiltonian can be shown as

$$H=J(\sigma_1^+\sigma_2^-+\sigma_1^-\sigma_2^+)+J\gamma(\sigma_1^+\sigma_2^++\sigma_1^-\sigma_2^-)+\frac{J_z}{2}\sigma_1^z\sigma_2^z+B_1\sigma_1^z+B_2\sigma_2^z, \tag{3}$$

where $J=\frac{J_x+J_y}{2}$, $\gamma=\frac{J_x-J_y}{J_x+J_y}$ and $\sigma^{\pm}=\frac{1}{2}(\sigma^x\pm i\sigma^y)$. Among these parameters $\sigma^{\pm}$ are raising and lowering operators respectively and $\gamma(0<\gamma<1)$ measures the anisotropy in the XY plane. The eigenvectors and eigenvalues of H are easily obtained as $H|\psi\rangle_{1,2}=(\frac{J_z}{2}\pm\mu)|\psi\rangle_{1,2}$, $H|\psi\rangle_{3,4}=(-\frac{J_z}{2}\pm\nu)|\psi\rangle_{3,4}$, where $|\psi\rangle_{1,2}=\frac{1}{\sqrt{2\mu(\epsilon\pm\mu)}}[J\gamma|00\rangle-(\epsilon\pm\mu)|11\rangle]$, $|\psi\rangle_{3,4}=\frac{1}{\sqrt{2\nu(\epsilon\pm\mu)}}[J|01\rangle-(\eta\pm\nu)|10\rangle]$, and $\mu=\sqrt{\epsilon^2+J^2\gamma^2}$, $\nu=\sqrt{\eta^2+J^2}$ with $\epsilon=B_1+B_2$, $\eta=B_1-B_2$. When $J\neq 0$, the eigenvectors represent entangled states, so it means that entanglement exists for both antiferromagnetic ($J > 0$) and ferromagnetic ($J < 0$) cases. Also, when $\epsilon = 0, \eta = 0$, these four eigenstates reduce to four Bell states that $|\psi\rangle_{1,2} \longrightarrow |\Psi\rangle^{\mp} = \frac{1}{\sqrt{2}}(|00\rangle \mp |11\rangle)$, $|\psi\rangle_{3,4} \longrightarrow |\Phi\rangle^{\mp} = \frac{1}{\sqrt{2}}(|01\rangle \mp |10\rangle)$.

Then, using the Hamiltonian (3), the square roots of the eigenvalues of the matrix R can be calculated as following

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{Z} e^{-\frac{J_z}{2T}} \left| \sqrt{1 + \frac{J^2\gamma^2}{\mu^2}\sinh^2\frac{\mu}{T}} \mp \frac{J\gamma}{\mu}\sinh\frac{\mu}{T} \right|, \\ \lambda_{3,4} &= \frac{1}{Z} e^{\frac{J_z}{2T}} \left| \sqrt{1 + \frac{J^2}{\nu^2}\sinh^2\frac{\nu}{T}} \mp \frac{J}{\nu}\sinh\frac{\nu}{T} \right|, \end{aligned} \tag{4}$$

where the partition function Z is given by

$$Z = 2\left(e^{-\frac{J_z}{2T}} \cosh\frac{\mu}{T} + e^{\frac{J_z}{2T}} \cosh\frac{\nu}{T} \right). \tag{5}$$

The λ_i are in arbitrary order. Since their relative magnitudes depend on the parameters involved, they cannot be ordered by magnitude unless the parameter values are known. For particular parameters, Concurrence C can be evaluated numerically or even analytically. The concurrence C derived from (3)–(5) is invariant under the substitutions $J \longrightarrow -J$, which indicates that the entanglement is the same for the antiferromagnetic and ferromagnetic cases. The concurrence is also the same for γ and $-\gamma$. Therefore, we restrict our considerations to $J > 0$ and $0 \le \gamma \le 1$.

3 The Result and Discussion

From (4) and the definition of the concurrence, we derive the C for $T = 0$ as following

$$C(T=0) = \begin{cases} \frac{J\gamma}{\mu}, & \nu < \mu - J_z, \\ |\frac{J\gamma}{\mu} - \frac{J}{\nu}|, & \nu = \mu - J_z, \\ \frac{J}{\nu}, & \nu > \mu - J_z. \end{cases} \tag{6}$$

Although the parameters J, γ, μ and ν are independent of J_z in the case of two interacting qubits at zero temperature, the value of J_z is very important in determining the point of the piecewise function so that it can play role in the pairwise entanglement. For the inhomogeneous external magnetic field case ($\epsilon = 0$) [15], there is a critical inhomogeneous magnetic field value η_c that is given by $\eta_c = \sqrt{(J\gamma - J_z)^2 - J^2}$, where the entanglement becomes a nonanalytic function of η and a quantum phase transition occurs. For $\eta < \eta_c$, the concurrence C is constant and equal to its initial value, but for $\eta > \eta_c$ the concurrence C undergoes a revival before decreasing to zero. In Ref. [15], the region of entanglement keeping its constant large value is broaden with the increasing of γ, that is, γ causes a shift in the locations of the phase transitions. For the homogeneous external magnetic field case ($\eta = 0$), there is also a critical value of ϵ_c which is given by $\epsilon_c = \sqrt{(J + J_z)^2 - J^2\gamma^2}$. From the expression of ϵ_c and η_c, we can calculate all of the condition in which the critical value can be increased, e.g., when we set $\eta = 0$, $J = 1$, $0 \le \gamma \le 1$, the critical homogeneous magnetic field ϵ_c is increased by the increasing of the interaction of z-component J_z.

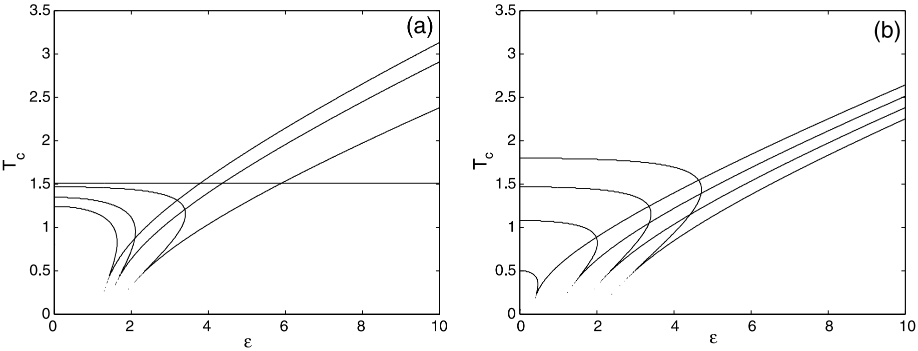

Fig. 1 Temperature at which the concurrence C vanishes, plotted as a function of the magnetic field strength ϵ for (**a**) $J = 1, \eta = 0, J_z = 0.5, \gamma = 0$ (transverse), $\gamma = 0.4, 0.8, 1$ (*from bottom to top*); (**b**) $J = 1, \gamma = 0.4, \eta = 0, J_z = 1, 0.5, 0, -0.5$ (*from bottom to top*)

Consider now the case $T > 0$, for which no quantum phase transition occurs [16]. In this case, C cannot be written analytically in as simple a form as (6); rather, one must obtain C numerically using (3)–(5). In every case as T increase, there exists for any external field a temperature above which the entanglement vanishes identically. This critical temperature T_c is independent of B in the isotropic case (Fig. 1(a)) and is given by $K_B T_c = 1.1346J$ [9]. In contrast to the isotropic case, T_c in the anisotropic case is a function of J, η, ϵ, J_z and γ. This is illustrated in Fig. 1(a, b), which are contour plots extracted from figures of concurrence vs ϵ and T in every corresponding cases. The curves in Fig. 1(a, b) show at what temperature C vanishes as a function of the external magnetic field and the regions under the curves indicate for $C > 0$. On each plot, when there are two values of T (for fixed ϵ) for which C vanishes, there is a revival of C as a function of T, with C vanishing above the higher value of T. Correspondingly, for fixed T, when there are two critical values of ϵ for which C vanishes, C is zero for ϵ between these two values; for ϵ above the second critical value, C undergoes a revival. As shown in Fig. 1(a), for sufficiently large ϵ ($\eta = 0$), T_c increases with both ϵ and γ. These results show that the anisotropy permits one to obtain entangled qubits at higher T and higher ϵ than is possible in the isotropic case. And there are revival phenomenons for all the $\gamma \neq 0$ cases, this contrasts with the anisotropic XY chain, for which there is no revival phenomenons for the $\gamma = 1$ case [11]. In Fig. 1(b), for fixed ϵ and γ, the critical temperature T_c increases with the decreasing of J_z, that is the critical temperature T_c is larger as we introduce negative parameter of J_z than the positive case. Above all for any finite T, there is a ϵ for which $C > 0$.

For a finite temperature T Ref. [17] pointed out that a pairwise entanglement in N-qubit isotropic Heisenberg system in certain degree can be increased by introducing external field B. Our study shows that entanglement exists in two regions: the one is in small γ and ϵ and in this region the entanglement is decreased with the increasing of γ and ϵ; the other is in large γ and ϵ where the entanglement is increased with the increasing of γ and ϵ, which is shown in Fig. 2(a). Figure 2(b) shows the concurrence C in terms of the inhomogeneity η and anisotropy parameter γ at a finite temperature T for $\epsilon = 1$, $Jz = -0.6$. Figure 2(c) shows the entanglement as measured by the concurrence in terms of the interaction of z-component Jz and anisotropy parameter γ at a finite temperature T for $\eta = 0$, $\epsilon = 0.8$. We can observe from Fig. 2(b, c) that entanglement exists in two regions of large one parameter and small the other parameter, that is to say if the anisotropy of XY plane and the interaction of z-component J_z are large enough, we just need small inhomogeneous

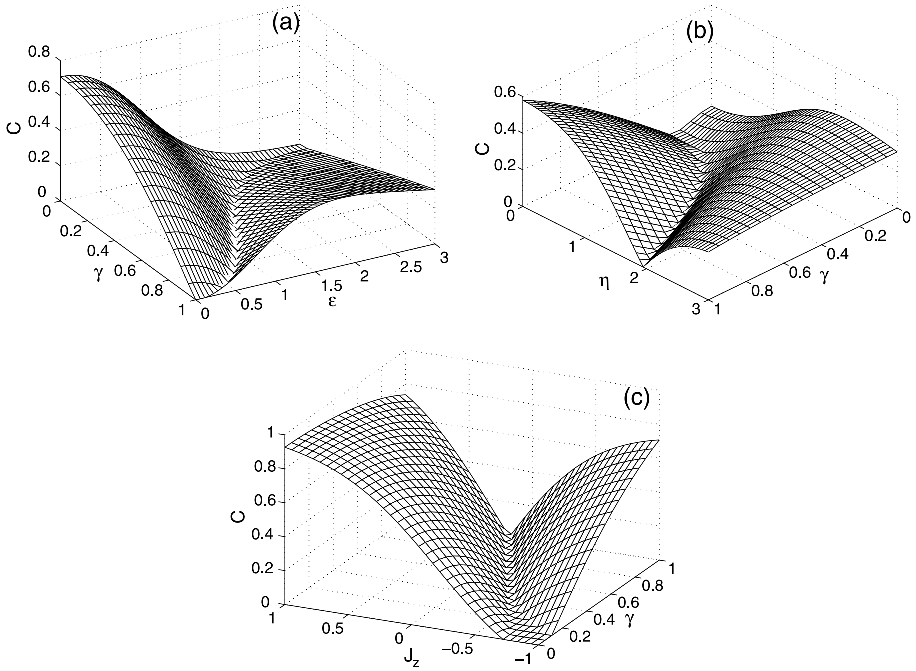

Fig. 2 Concurrence in the two-qubit Heisenberg XYZ chain is plotted for $T=0.4$, $J=1$. (**a**) Concurrence vs γ and ϵ for $\eta=0$, $Jz=0$; (**b**) Concurrence vs γ and η for $\epsilon=1$, $Jz=-0.6$; (**c**) Concurrence vs γ and Jz for $\eta=0$, $\epsilon=0.8$

external field η or even without it, vice versa. At this point, the role of η is something similar to the anisotropy of XY plane and J_z. It seems to be possible to control the anisotropy and the interaction of z-component J_z by adjusting the inhomogeneity of the external magnetic field.

4 Conclusion

In conclusion, our study of the anisotropic two-qubit XYZ model in an inhomogeneous external magnetic field reveals the strong combined influence of the anisotropy parameter γ and the magnetic field on the entanglement at thermal equilibrium. Through analyzing the $T=0$ case, we find that conditions to broaden the regions of the entanglement keeping it's constant large values. We can modulate the concurrence at a finite temperature in regions of (γ,ϵ), (γ,η), (γ, J_z) parameter space. Most importantly, we provide the means to produce entanglement in two spin systems for any finite T by adjusting the external inhomogeneous magnetic field.

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